Exercise 3B

1 a Let $y = \text{arcosh}\,2x$ then $\cosh y = 2x$

Differentiate with respect to x

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \quad \text{but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

b Let $y = \text{arsinh}(x+1)$ then $\sinh y = x+1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \quad \text{but } \sinh y = x+1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

c Let $y = \text{artanh}\,3x$

$$\tanh y = 3x$$

$$\text{sech}^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\text{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - \tanh^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - 9x^2}$$

1 **d** Let $y = \text{arsech}x$

$\text{sech}y = x$

$\frac{1}{\cosh y} = x$

$1 = x\cosh y$

Differentiate with respect to x

$$0 = \cosh y + x\sinh y\frac{dy}{dx}$$

$$x\sinh y\frac{dy}{dx} = -\cosh y$$

$$\frac{dy}{dx} = \frac{-\cosh y}{x\sinh y}$$

$$= \frac{-1}{x\tanh y}$$

$$= \frac{-1}{x(1-\text{sech}^2 y)^{\frac{1}{2}}}$$

$$= \frac{-1}{x(1-x^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{x^2\sqrt{\frac{1}{x^2}-1}}$$

e Let $y = \text{arcosh}\, x^2$

Let $t = x^2$ $\quad y = \text{arcosh}\, t$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{\sqrt{t^2-1}}\right)(2x)$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$$

f $y = \text{arcosh}\, 3x$

Let $t = 3x$ $\quad y = \text{arcosh}\, t$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{\sqrt{t^2-1}}\right)(3)$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2-1}}$$

g $y = x^2\text{arcosh}\, x$

$$\frac{dy}{dx} = 2x\text{arcosh}\, x + \frac{x^2}{\sqrt{x^2-1}}$$

1 h $y = \text{arsinh}\,\frac{x}{2}$

Let $t = \frac{x}{2}$ $y = \text{arsinh}\,t$

$\frac{dt}{dx} = \frac{1}{2}$ $\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$

$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{1}{\sqrt{t^2+1}}\left(\frac{1}{2}\right)$

$= \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2+1}} = \frac{1}{\sqrt{x^2+4}}$

i $y = e^{x^3}\text{arsinh}\,x$

$\frac{dy}{dx} = 3x^2e^{x^3}\text{arsinh}\,x + \frac{e^{x^3}}{\sqrt{x^2+1}}$

j $y = \text{arsinh}\,x\,\text{arcosh}\,x$

$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\text{arcosh}\,x + \frac{1}{\sqrt{x^2-1}}\text{arsinh}\,x$

k $y = \text{arcosh}\,x\,\text{sech}\,x$

$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}\text{sech}\,x - \text{arcosh}\,x\tanh x\,\text{sech}\,x$

$= \text{sech}\,x\left(\frac{1}{\sqrt{x^2-1}} - \text{arcosh}\,x\tanh x\right)$

l $y = x\,\text{arcosh}\,3x$

$\frac{dy}{dx} = \text{arcosh}3x + x \times \frac{3}{\sqrt{9x^2-1}}$

$\frac{dy}{dx} = \text{arcosh}3x + \frac{3x}{\sqrt{9x^2-1}}$

2 a Let $y = \text{arcosh}x$

$\cosh y = x$

$\sinh y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sinh y}$

$= \frac{1}{\sqrt{\cosh^2 y - 1}}$

$= \frac{1}{\sqrt{x^2-1}}$ as required

2 b Let $y = \operatorname{artanh} x$

$\tanh y = x$

$\operatorname{sech}^2 y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

$$= \frac{1}{1 - \tanh^2 y}$$

$$= \frac{1}{1 - x^2} \text{ as required}$$

3 $y = \operatorname{artanh} \dfrac{e^x}{2}$

Let $t = \dfrac{e^x}{2}$ $\quad y = \operatorname{artanh} t$

$\dfrac{dt}{dx} = \dfrac{e^x}{2}$ $\quad \dfrac{dy}{dt} = \dfrac{1}{1 - t^2}$

Then $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right)\left(\dfrac{dt}{dx}\right) = \dfrac{1}{1 - t^2} \times \dfrac{e^x}{2}$

$$= \frac{1}{1 - \left(\frac{e^x}{2}\right)^2} \times \frac{e^x}{2}$$

$$= \frac{\frac{e^x}{2}}{\frac{4 - e^{2x}}{4}}$$

$$\frac{dy}{dx} = \frac{2e^x}{4 - e^{2x}}$$

$$(4 - e^{2x})\frac{dy}{dx} = 2e^x$$

4 $y = \operatorname{arsinh} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+1)^{-\frac{3}{2}}2x$$

$$= \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{-1(x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} \times 2x \times -x}{(x^2+1)^3}$$

$$= \frac{3x^2(x^2+1)^{\frac{1}{2}} - (x^2+1)^{\frac{3}{2}}}{(x^2+1)^3}$$

$$= \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} - \frac{1}{(x^2+1)^{\frac{3}{2}}}$$

$$(x^2+1)\frac{d^3y}{dx^3} = \frac{3x^2}{(x^2+1)^{\frac{3}{2}}} - \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

$$= -3x\frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

5 $y = (\operatorname{arcosh} x)^2$

$$\frac{dy}{dx} = 2\operatorname{arcosh}x \times \frac{1}{\sqrt{x^2-1}}$$

$$= 2(x^2-1)^{-\frac{1}{2}}\operatorname{arcosh} x$$

$$\frac{d^2y}{dx^2} = -(x^2-1)^{-\frac{3}{2}}2x\operatorname{arcosh} x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}}$$

$$= \frac{-2x\operatorname{arcosh} x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}$$

6 $y = \operatorname{artanh} x \quad x = \frac{12}{13} \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln 25 = \ln 5$

$$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$$

Tangent is

$$(y - \ln 5) = \frac{169}{25}\left(x - \frac{12}{13}\right)$$

$$25y - 25\ln 5 = 169x - 156$$